Control Systems I

Stability

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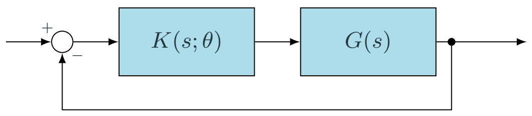
Today : Stability of Closed-Loop System



Question: For what controllers K is the closed-loop stable?

Nyquist cirterion: Simple method to determine if the closed-loop system will be stable by looking at the ***open-loop*** Nyquist plot.

- Can generate the open-loop Nyquist plot from system measurements without doing any modeling
- Can easily define a range of stabilizing control gains
- Tells us how close the system is to unstable - will use to define the notion of robustness next week

Stability - Example

Video : f22

Stability - Example

Video : tacoma

BIBO Stability

Bounded Input Bounded Output (BIBO)

- If the input signal is bounded energy, then the system will not cause the output to have unbounded energy

A signal $w(t)$ is said to be ***bounded*** if there exists a constant C such that $|w(t)| \leq C$ for all $t \geq 0$

BIBO Stable System

A system is said to be BIBO stable, if the output signal is bounded for all bounded input signals.

- Stability is a basic property that all closed-loop systems must have
- Note that a BIBO stable system does not guarantee that the system does anything *useful*, only that it doesn't explode!
- There are many more types of stability

Stability Condition

Theorem

An LTI system is BIBO stable if its impulse response is absolutely integrable

$$\int_0^\infty |g(t)| < \infty$$

$$\begin{aligned} y(t) &= \int_{\tau=0}^{t} u(\tau) g(t-\tau) d\tau \\ &= \int_{\tau=0}^{t} g(\tau) u(t-\tau) d\tau \end{aligned}$$

Assume bounded input $u(t) \leq C$

$$\begin{aligned} |y(t)| &\leq \int_{\tau=0}^{t} |g(\tau) u(t-\tau)| d\tau \\ &\leq C \int_{\tau=0}^{t} |g(\tau)| d\tau \end{aligned}$$

and we see that y is bounded if g is absolutely integrable

Stability Condition

Stability via System Poles

An LTI system with transfer function $G(s)$ is BIBO stable if and only if all its poles are in the left half plane.

Transfer function

$$G(s) := \frac{\prod_{i=0}^{n_z}(s-z_i)}{\prod_{j=0}^{n_p}(s-p_i)} = \frac{c_1}{s-p_1} + \cdots + \frac{c_{n_p}}{s-p_{n_p}}$$

In the time-domain:

$$g(t) = c_1 e^{p_1 t} + \cdots + c_{n_p} e^{p_{n_p} t}$$

We see that $g(t)$ is integrable only if $\mathrm{Re}(p_i) < 0$ for all i.

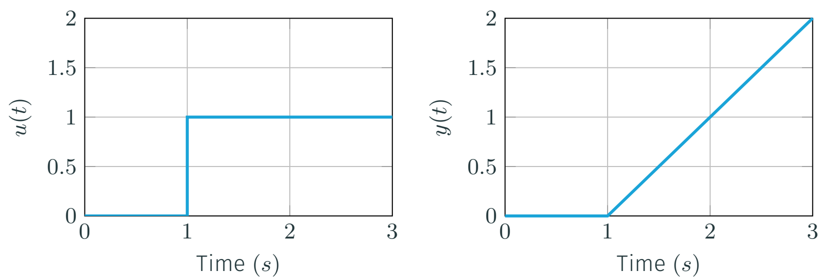
Note: This idea extends to systems with poles at zero, and multiple poles.

Example - Integrator

$$\dot{y} = u \qquad\qquad G(z) = \frac{1}{s}$$

Not BIBO stable, because pole is on the imaginary axis.

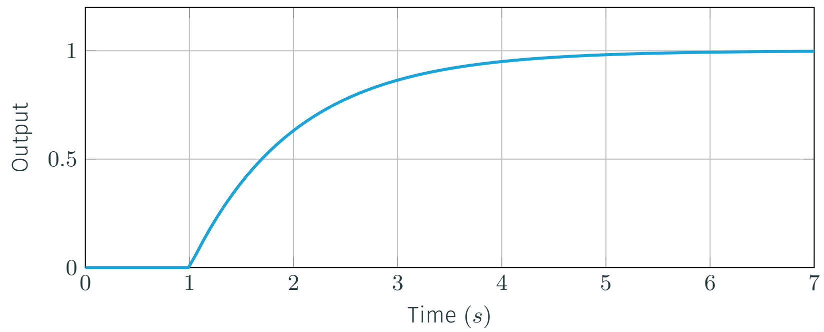
Response to the bounded step input $u(t) = 1, t > 0$, $y = t$



Example - Delayed First-Order System

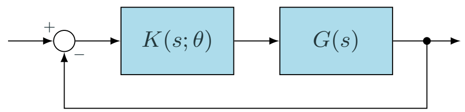
$$G(s) = \frac{e^{-s}}{s+1}$$

All the poles are in the left half plane - stable.



Nyquist Criterion

Stability of Closed-Loop System



Suppose we know $G(s)$, and a ***parameterized*** version of our controller $K(s;\theta)$

→ For what values of θ is the closed-loop stable?

Example 1: PI

$$K(s) = K_p\left(\frac{1}{T_i s} + 1\right) \qquad \text{parameters: } K_p, T_i$$

Example 2: Stabilizing controllers

$$K(s) = \frac{a(s)}{b(s)} \qquad \text{poles of } K(s) \text{ in the LHP}$$

Nyquist Criterion

Goal: Decide if closed-loop is stable by looking at open-loop Nyquist diagram

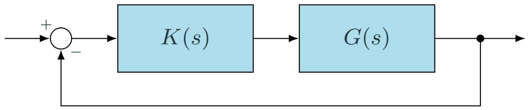
Why?

- Simple test to know if a controller will be stable without computing the closed-loop poles
- We can ***shape*** the open-loop behaviour as desired by changing the controller in $K(s)G(s)$
- Simple design methods based on *experimental data*
- Determine ***robustness*** of the closed-loop system to uncertain model parameters, noise, structure, etc

⋮

- Optimize *performance* over all possible control laws subject to *stability* of closed-loop system

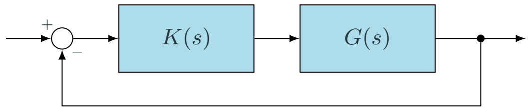
Closed-Loop Transfer Functions



Controller K and system H are rational functions of polynomials in s

$$K(s) = \frac{S(s)}{R(s)} \qquad G(s) = \frac{B(s)}{A(s)}$$

Closed-Loop Transfer Functions



Controller K and system H are rational functions of polynomials in s

$$K(s) = \frac{S(s)}{R(s)} \qquad G(s) = \frac{B(s)}{A(s)}$$

So the closed-loop system is:

$$G_{cl}(s) = \frac{K(s)G(s)}{1 + K(s)G(s)} = \frac{B(s)S(s)}{A(s)R(s) + B(s)S(s)}$$

Stability of Closed-Loop System

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Closed-loop system is stable if and only if the *roots of the characteristic polynomial are in the left half plane*

$$p(s) := A(s)R(s) + B(s)S(s)$$

True if and only if the zeros of $1 + K(s)G(s)$ are in the left half plane (LHP)

Stability of Closed-Loop System

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True if and only if the zeros of $1 + K(s)G(s)$ are in the left half plane (LHP)

$$1 + K(s)G(s) = 1 + \frac{S(s)}{R(s)} \cdot \frac{B(s)}{A(s)} = \frac{A(s)R(s) + B(s)S(s)}{A(s)R(s)}$$

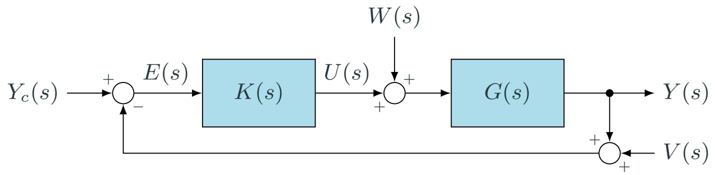
Notes

$$G_{cl}(s) = \frac{K(s)G(s)}{1 + K(s)G(s)} = \frac{B(s)S(s)}{A(s)R(s) + B(s)S(s)}$$

- The closed-loop poles are totally different from the open-loop ones
- The closed-loop system contains the open-loop zeros $S(s)$
- All four basic transfer functions have the same poles

$$\frac{G(s)K(s)}{1+G(s)K(s)} \qquad \frac{1}{1+G(s)K(s)} \qquad \frac{G(s)}{1+G(s)K(s)} \qquad \frac{K(s)}{1+G(s)K(s)}$$

All Transfer Functions



Transfer functions from all inputs to all outputs:

$$\begin{bmatrix} Y'(s) \\ E(s) \\ U'(s) \end{bmatrix} = \frac{1}{1+G(s)K(s)} \begin{bmatrix} G(s)K(s) & G(s) & 1 \\ 1 & -G(s) & -1 \\ K(s) & 1 & -K(s) \end{bmatrix} \begin{bmatrix} Y_c(s) \\ W(s) \\ V(s) \end{bmatrix}$$

Four basic transfer functions:

$$\frac{G(s)K(s)}{1+G(s)K(s)} \qquad \frac{1}{1+G(s)K(s)} \qquad \frac{G(s)}{1+G(s)K(s)} \qquad \frac{K(s)}{1+G(s)K(s)}$$

- Functions have the same *poles* or fundamental *modes*
- All depend on the ***open-loop transfer function*** $G(s)K(s)$

Nyquist Criterion : The Idea

What we want to know:

- Does $1 + K(s)G(s)$ have any zeros in the right-half plane?

The information we have:

- A plot of $K(s)G(s)$ for $s = j\omega$, i.e., on the imaginary axis
  (Note that this is a closed D-shaped contour around the RHP)

The trick from complex analysis:

- The number of poles / zeros of $1 + K(s)G(s)$ outside the LHP is equal to the number of times the curve $1 + K(j\omega)H(j\omega)$ encircles the origin

We can tell if the closed-loop system is stable by counting how many times the nyquist plot of the open-loop system encircles the origin.

Cauchy's Argument Principle

Consider the function:

$$H_1(s) := \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$$

Evaluate $H_1(s)$ at a point s_0, and write in polar notation

$$H_1(s_0) = \frac{(s_0 - z_1)(s_0 - z_2)}{(s_0 - p_1)(s_0 - p_2)} = \frac{r_{z_1} e^{j\theta_1} r_{z_2} e^{j\theta_2}}{r_{p_1} e^{j\phi_1} r_{p_2} e^{j\phi_2}}$$

we see that the argument is additive

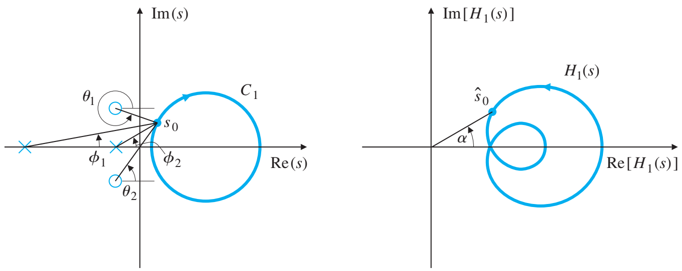
$$\angle H_1(s_0) = \theta_1 + \theta_2 - (\phi_1 + \phi_2)$$

where θ_i, ϕ_i is the angle from the i^{th} pole/zero to the point s_0

Cauchy’s Argument Principle

Let the point s_0 follow a smooth, closed, non-self-intersecting curve C_1

Case 1: No poles or zeros inside C_1



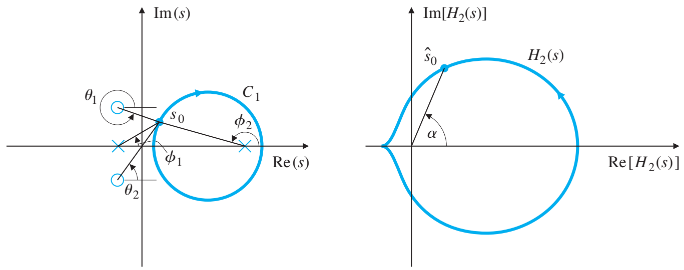
$$\angle H_1(s) = \alpha = \theta_1 + \theta_2 - (\phi_1 + \phi_2)$$

- α will increase and decease as s changes, but integral of α around C_1 is zero
- Implication: The curve $\{H_1(s) \mid s \in C_1\}$ does not contain the origin

Cauchy’s Argument Principle

Let the point s_0 follow a smooth, closed, non-self-intersecting curve C_1

Case 2: Pole inside C_1



$$\angle H_1(s) = \alpha = \theta_1 + \theta_2 - (\phi_1 + \phi_2)$$

- Integral of α around C_1 is $-360°$
- Implication: The curve $\{H_1(s) \mid s \in C_1\}$ contains the origin
- (Note: A zero would cause an increase of $360°$)

Cauchy’s Argument Principle

Cauchy’s Argument Principle

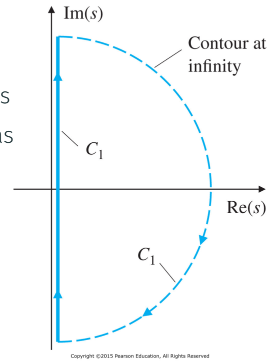
A contour map of a complex function will encircle the origin $Z - P$ times, where Z is the number of zeros and P is the number of poles of the function inside the contour.

How to use this for control?

Nyquist Plots and Cauchy’s Argument Principle

The question: Does $1 + K(s)G(s)$ have a zero in the right half plane?

- Take the contour to be a clockwise encirclement of the right half plane
- Assume for now that $K(s)G(s)$ has no unstable poles
- The Argument Principle tells us that $1 + K(s)G(s)$ has a zero in the right half plane if the plot $1 + K(j\omega)G(j\omega)$ contains the origin
- This is equivalent to saying that the Nyquist plot of $K(s)G(s)$ does not contain the point -1



Nyquist Criteria

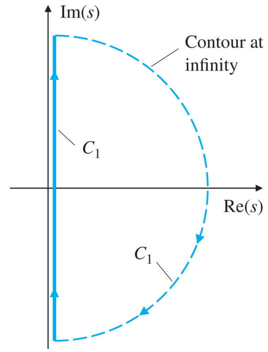
1. Plot the Nyquist plot of $K(s)G(s)$
2. Evaluate the number of clockwise encirclements of -1, call this N
   - Draw a straight line in any direction from -1 to ∞
   - Count the net number of left-to-right crossings of the straight line by $K(s)G(s)$. This is N
   - Right-to-left crossing decrease N by one. N can be negative.
3. Determine the number of unstable poles of $G(s)$, call this P
4. Calculate the number of unstable closed-loop roots Z

$$Z = N + P$$

Nyquist criterion: The closed loop system is stable if and only if Z is zero.

Detail : Why can we plot only the imaginary axis?

We should plot the full D-contour, but we only plot the contour along the imaginary axis. Why?

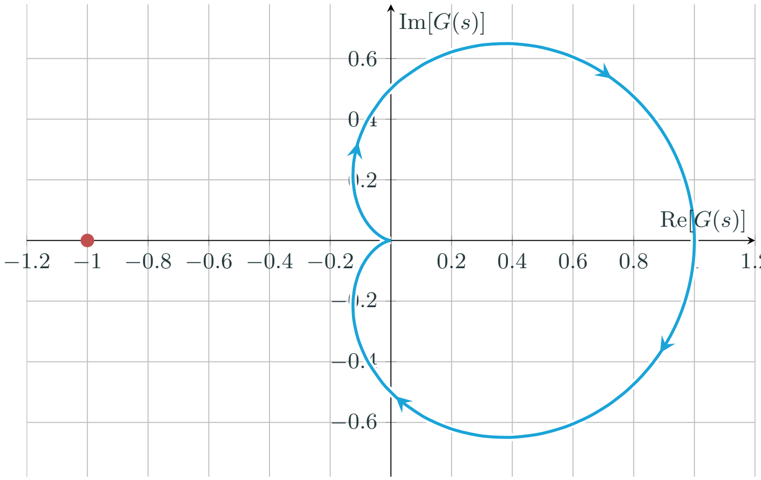


All physical systems have more poles than zeros (strictly proper). This is because infinite frequency oscillations result in no motion (infinite acceleration is not possible).

$$\lim_{\omega\to\infty} |G(j\omega)| = 0 \qquad \rightarrow \qquad \text{Nyquist plot for the 'D' section is zero.}$$

Note: $G(-j\omega) = \bar{G}(j\omega)$. The Nyquist plot is symmetric around the real axis.

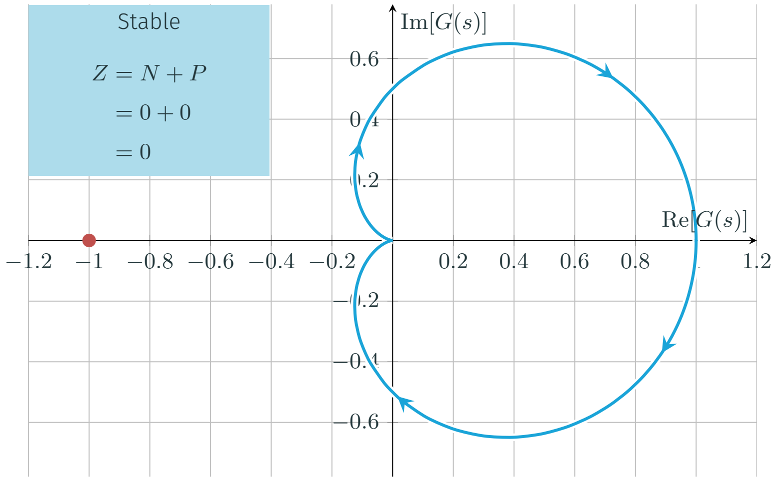
Example $G(s) = \frac{1}{(s+1)^2}$ $K(s) = 1$



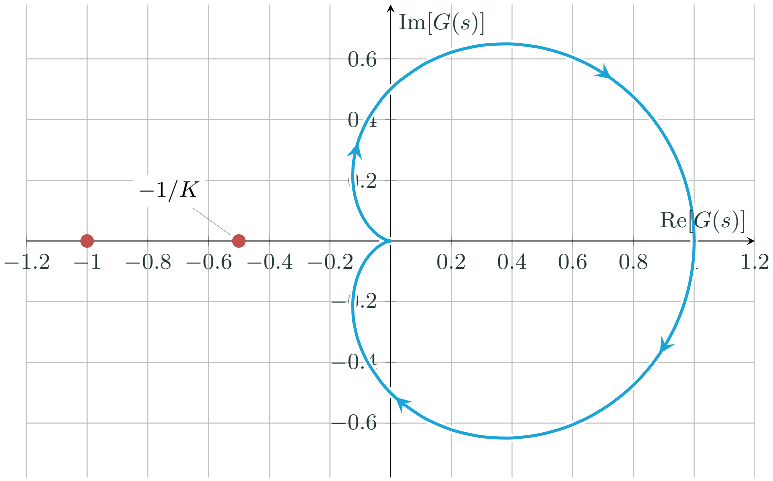
Example $G(s) = \frac{1}{(s+1)^2}$ $K(s) = 1$

Stable

$$Z = N + P$$
$$= 0 + 0$$
$$= 0$$

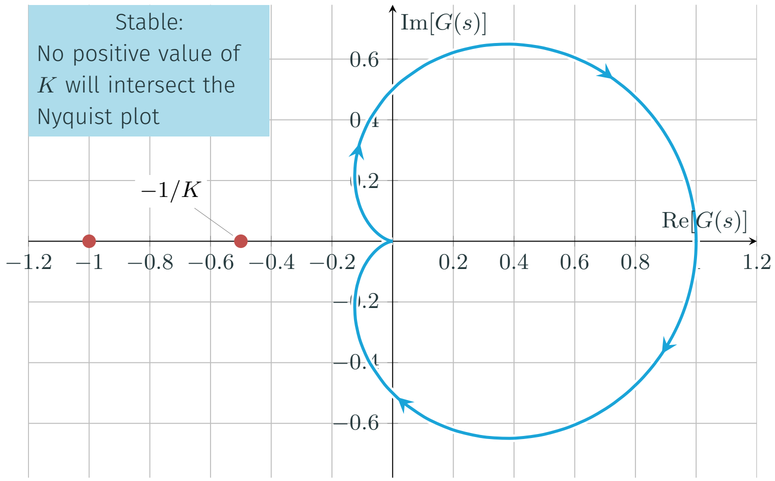


Example $G(s) = \frac{1}{(s+1)^2}$ $K(s) = 2 \leftarrow$ Still stable?

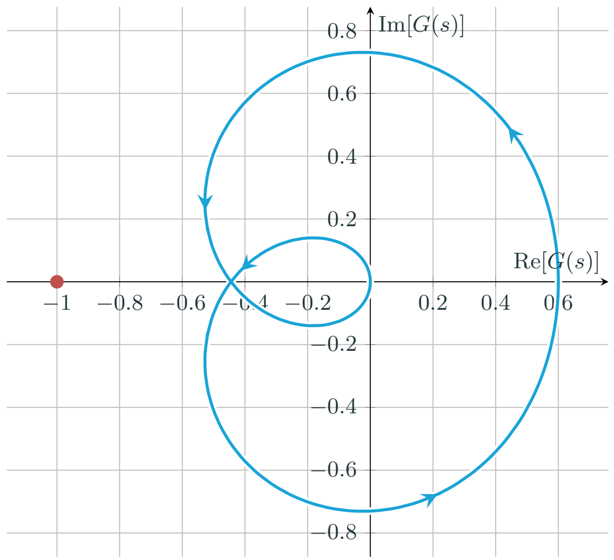


Example $G(s) = \frac{1}{(s+1)^2}$ $K(s) = 2 \leftarrow$ Still stable?

Stable:
No positive value of K will intersect the Nyquist plot



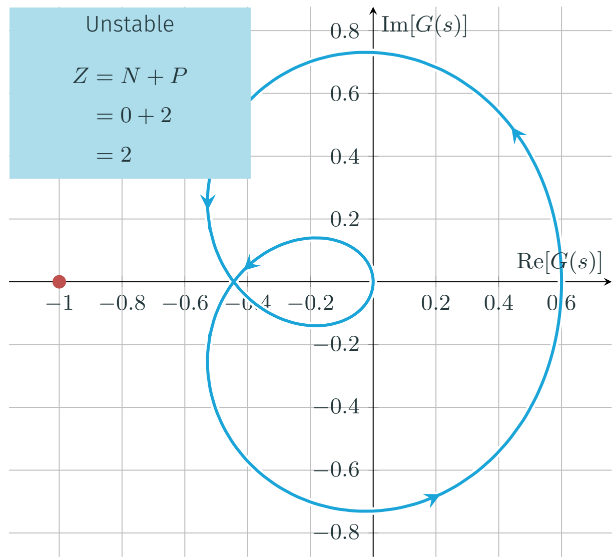
Example $G(s) = 0.4\frac{s+0.9}{s^2-0.9s+0.6}$ $K(s) = 1$



Example $G(s) = 0.4\frac{s+0.9}{s^2-0.9s+0.6}$ $K(s) = 1$

Unstable

$$\begin{aligned} Z &= N + P \\ &= 0 + 2 \\ &= 2 \end{aligned}$$



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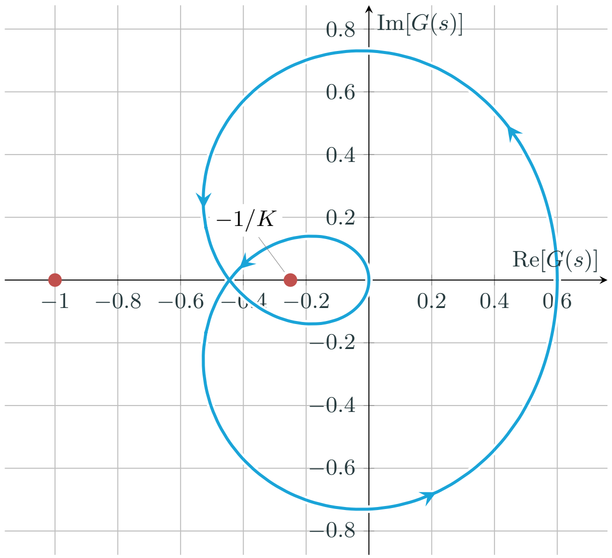

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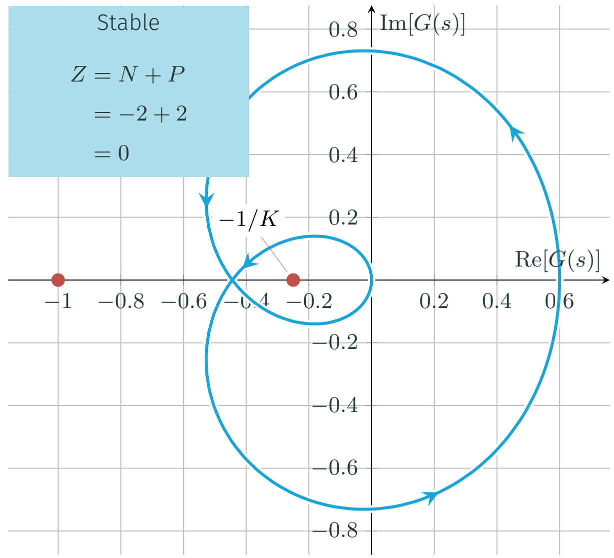

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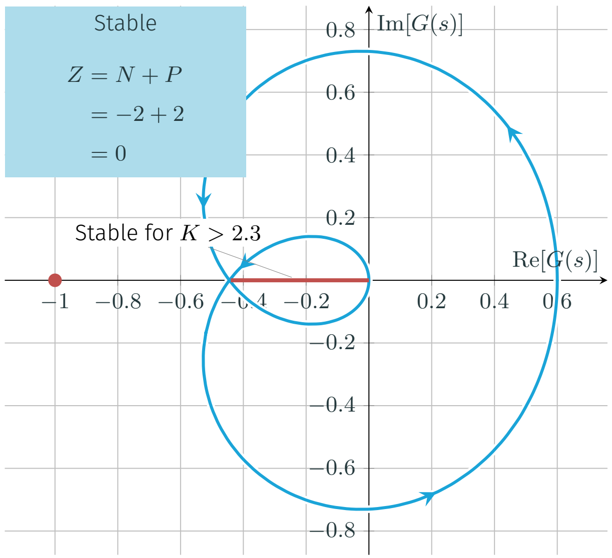
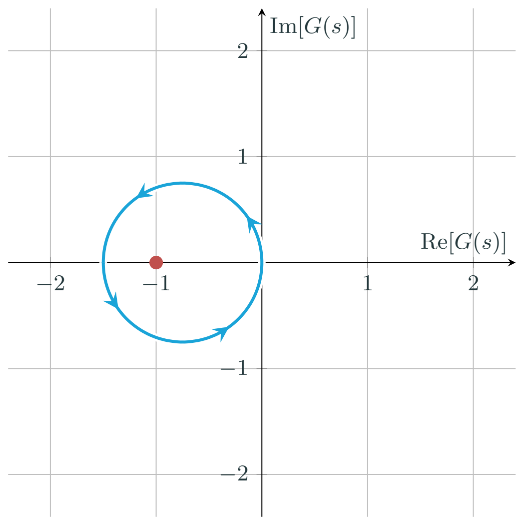


Example $\quad G(s) = \dfrac{1.5}{s-1} \qquad K(s) = 1$



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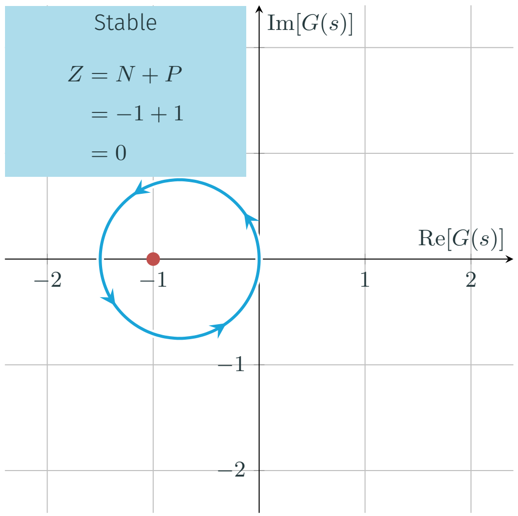
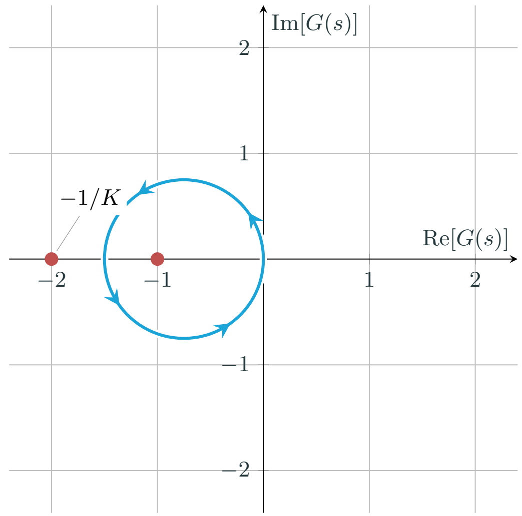


Example $G(s) = \frac{1.5}{s-1}$ $K(s) = \frac{1}{2}$



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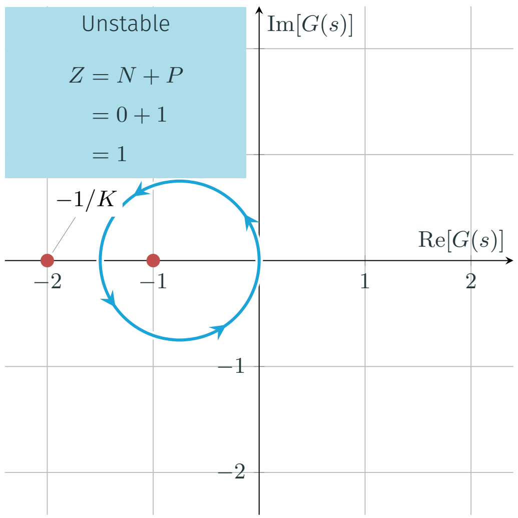

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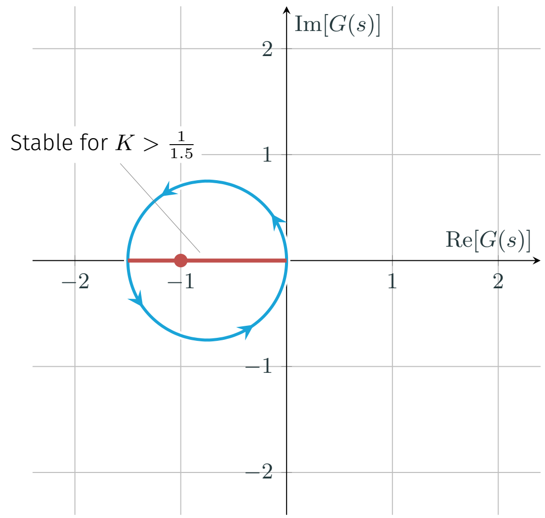
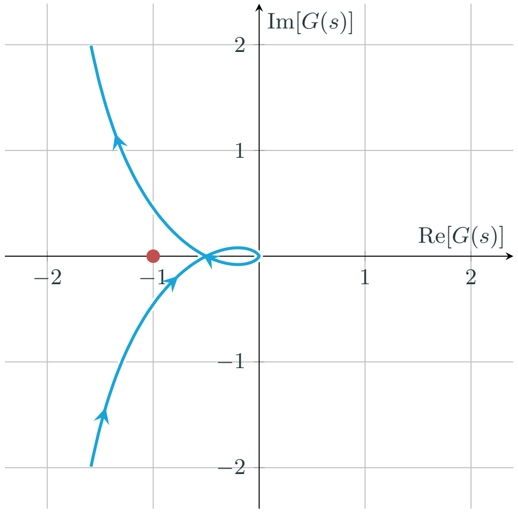


Example $G(s) = \frac{1}{s(s+1)^2}$ $K(s) = 1$



The Nyquist curve is unbounded as $\omega \to 0$.
Does the curve enclose the RHP, or the LHP?

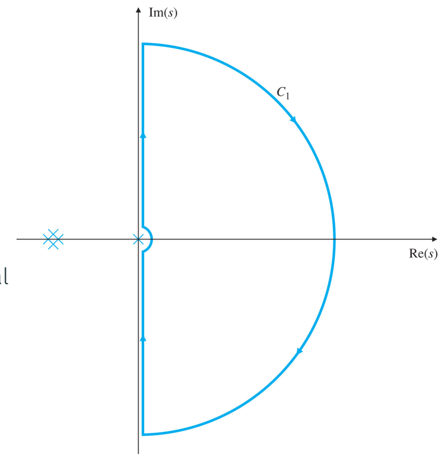
Impact of Integrators

Suppose our system has the form

$$K(s)G(s) = \frac{B(s)}{s^q A(s)}$$

What does the Nyquist plot look like?

Follow a curve that takes an infinitesimal curve around the point $s = 0$.

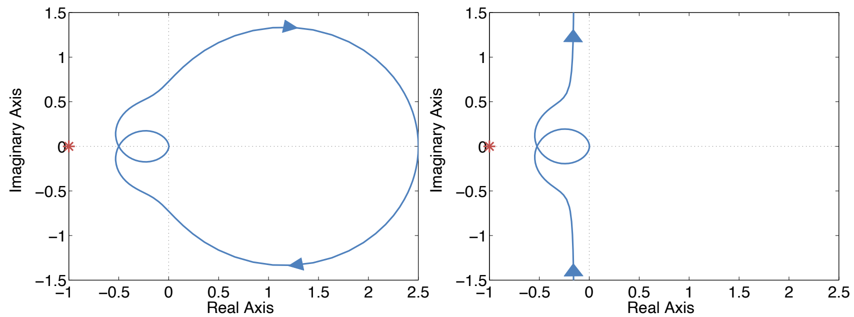


Impact of Integrators

'Integrator' Segment

$$s = re^{j\theta} \qquad K(s)G(s) = \frac{B(0)}{(re^{j\theta})^q A(0)} = \frac{\gamma}{r^q} e^{-jq\theta}, \quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

- $\gamma := B(0)/A(0)$ is the steady-state gain
- The Nyquist plot will form a semi-circle with infinite radius
- If $q > 1$, then we'll have multiple semi-circles



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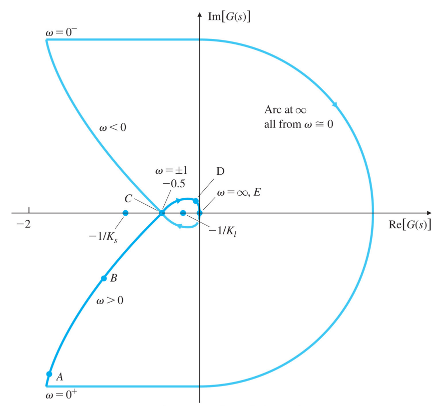
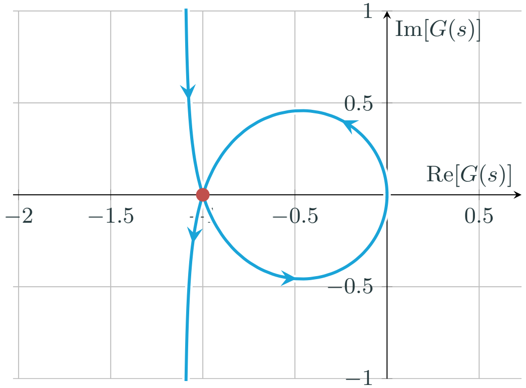


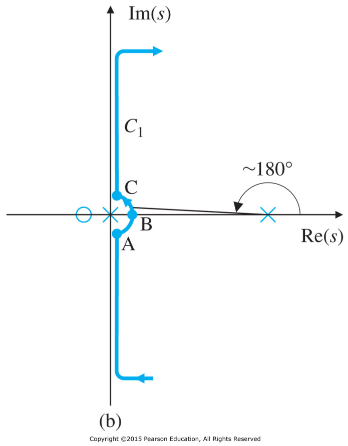
- $0 < K < 2$: Zero crossings, zero unstable open-loop poles $\to$ Stable
- $K > 2$: $N = 2$, $P = 0 \to Z = 2$
- $K < 0$: $N = 1$, $P = 0 \to Z = 1$

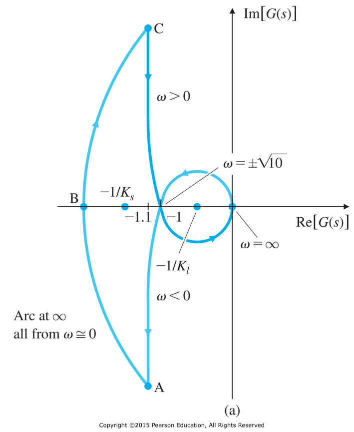
Example $G(s) = \dfrac{(s+1)}{s(s/10-1)}$ $K(s) = 1$



Stability depends on whether the infinite curve is clockwise or counterclockwise

Example $G(s) = \dfrac{(s+1)}{s(s/10-1)}$ $K(s) = 1$





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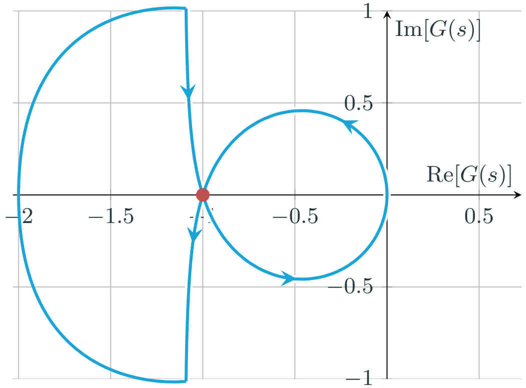

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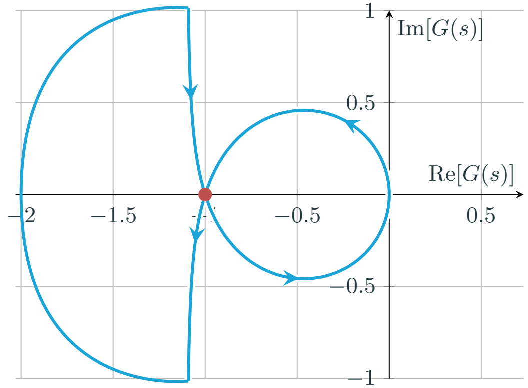


Stable for $K > 1$

Simplified Nyquist Criterion

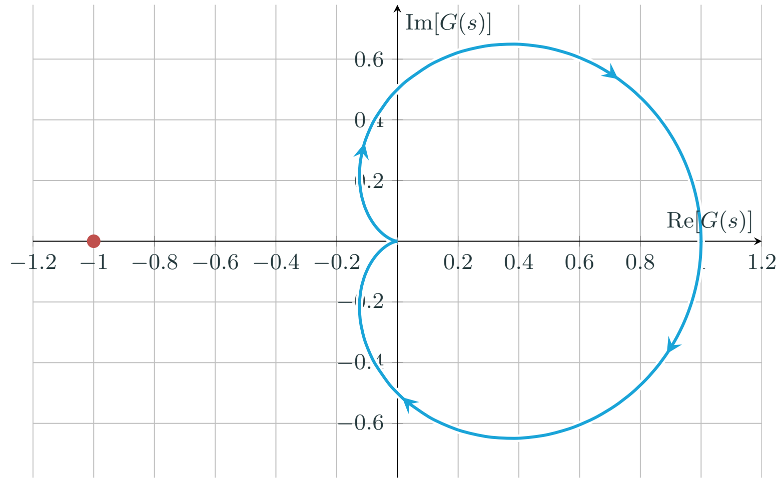
If all the poles of the system are stable, then there is a simpler condition.

- No unstable open-loop poles $\rightarrow P = 0$
- Number of unstable closed-loop poles $Z = N$, the number of encirclements

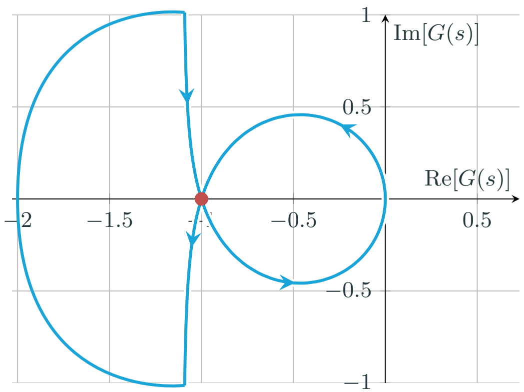
Simplified Nyquist Criterion

If the open-loop system is stable and the -1 point lies to the left of the Nyquist curve, then the closed-loop system is stable.

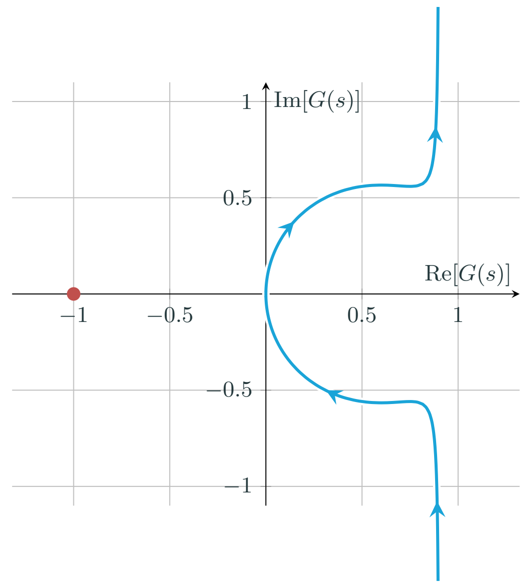
Example $\quad G(s) = \dfrac{1}{(s+1)^2} \quad K(s) = 1$



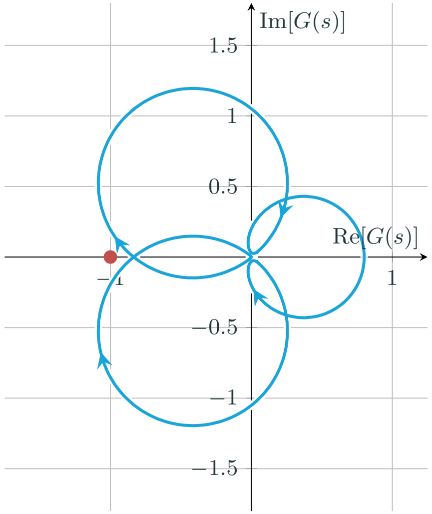
Example $\quad G(s) = \dfrac{(s+1)}{s(s/10-1)} \quad K(s) = 1$



Example $\quad G(s) = \dfrac{(s+1)}{s(s/10+1)} \quad K(s) = 1$

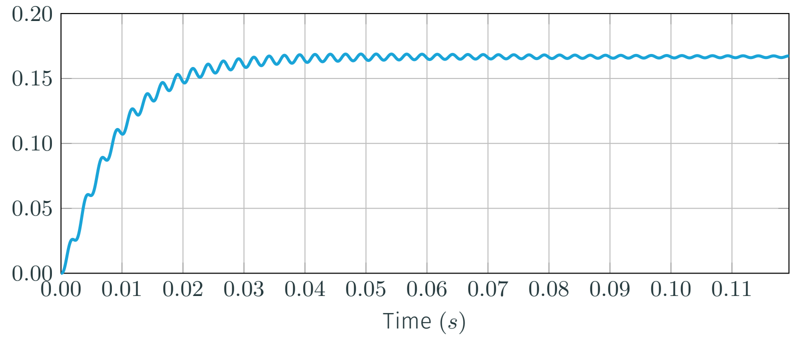


AFM $G(s) = \dfrac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s+3000)(s+1000)(s+100)(s^2+50s+6.25 \cdot 10^6)}$



AFM - Step response

Close-loop step response

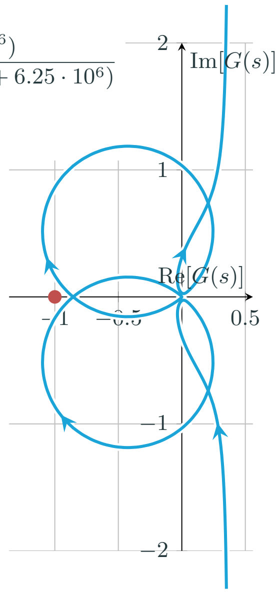


Steady-state offset $\rightarrow$ need an integrator!

AFM - PI Control

$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s+3000)(s+1000)(s+100)(s^2+50s+6.25 \cdot 10^6)}$$

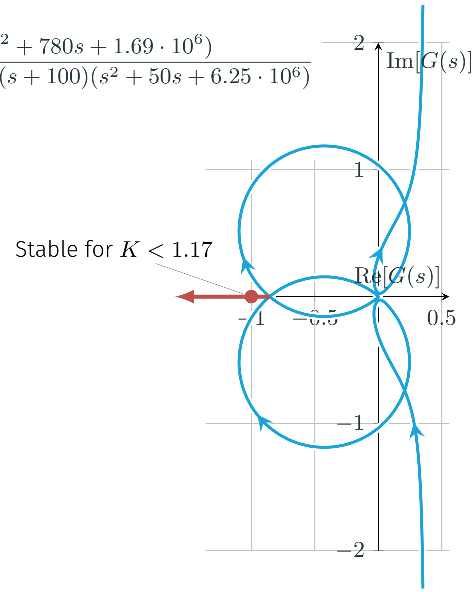
$$K(s) = K\left(1 + \frac{1}{T_i s}\right)$$



AFM - PI Control

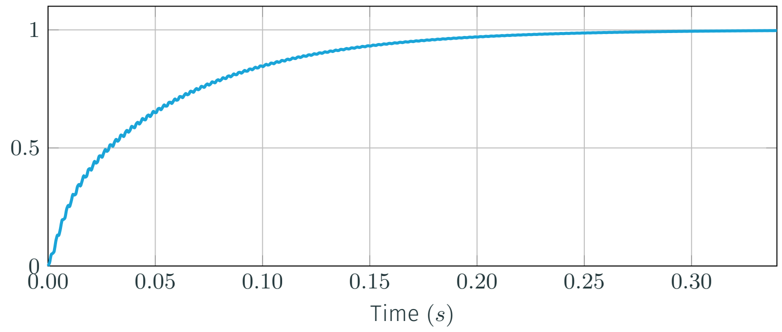
$$G(s) = \frac{8.88 \cdot 10^8 (s^2 + 780s + 1.69 \cdot 10^6)}{(s+3000)(s+1000)(s+100)(s^2+50s+6.25 \cdot 10^6)}$$

$$K(s) = K\left(1 + \frac{1}{T_i s}\right)$$



AFM - PI Control

Step response with $K = 0.5$



Summary

Nyquist cirterion: Simple method to determine if the closed-loop system will be stable by looking at the open-loop Nyquist plot.

Why?

- Can generate the open-loop Nyquist plot from system measurements without doing any modeling
- Can easily define a range of stabilizing control gains
- Tells us how close the system is to unstable - will use to define the notion of ***robustness*** next week